

Intermediate Macroeconomics: Consumption

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Fall 2012

1 Introduction

Consumption is the largest expenditure component in the US economy, accounting for between 60-70 percent of total GDP. In this set of notes we study consumption decisions. In micro you probably studied how people choose consumption among different goods in the cross-section: for example, how many apples and oranges to consume. In macro we study consumption in the time series dimension: how much total consumption does one do today versus in the future.

So as to study the behavior of consumption as a whole in the time series dimension, we engage in the fiction that households only consume one good. I will consistently refer to this one good throughout the course as “fruit,” though in reality it is more like a composite good or a basket of goods. Think about it this way – a household has some income to spend each period, and it must decide how much of that income to spend on consumption goods. We are going to study that decision. How that expenditure is split among different types of goods (e.g. apples and oranges) is the purview of microeconomists.

2 The Basic Two Period Model

Assume that a household lives for two periods: the present (t) and the future ($t + 1$). This is a useful abstraction to a multi-period horizon. The household has an exogenous stream of income in the two periods: Y_t and Y_{t+1} . We abstract from any uncertainty, so that Y_{t+1} is known at time t . The household begins life (period t) with no existing assets, though it would be straightforward to modify the environment to allow for that. The household can consume each period, C_t and C_{t+1} . It can also save or borrow in the first period, $S_t = Y_t - C_t$ (borrowing is negative saving). It earns/pays interest r_t on saving/borrowing, so that S_t today yields $(1 + r_t)S_t$ in income tomorrow.¹

Everything here is in real terms, which means that everything (including the real interest rate, r_t) is denominated in physical units of goods. It is helpful to think about income and consumption as being in the same units, and I like to use the fruit analogy. A household has an exogenous stream

¹Since the household effectively dies after period $t + 1$, it will not choose to do any saving in $t + 1$.

of fruit available to it each period; this is its income. C_t and C_{t+1} is how much fruit it actually eats each period. If it chooses to not consume some of its fruit in period t , so that $S_t > 0$, it can enter into a financial contract in which it gives up its fruit today in return for $(1 + r_t)S_t$ units of fruit tomorrow. In contrast, if it wants to consume more fruit today than it has, it can borrow some extra fruit, with $S_t < 0$, and will have to pay back $(1 + r_t)S_t$ units of fruit to the lender in period $t + 1$. The fruit is not storable on its own – if the household wants to save some of its fruit to eat tomorrow, it has to “put it in the bank” and earn r_t . Finally, the household is a price-taker: it takes r_t as given, and does not behave in any strategic way to try to influence r_t . Thus, from the household’s perspective r_t is exogenous, though from an economy-wide perspective (as we will see), it is endogenous.

The household thus faces two budget constraints: one in period t , and one in period $t + 1$, which I assume hold with equality:

$$\begin{aligned} C_t + S_t &= Y_t \\ C_{t+1} &= Y_{t+1} + (1 + r_t)S_t \end{aligned}$$

These two budget constraints can be combined into one: you can solve for S_t from either the first or the second period constraint, and then plug into the other one. Doing so, I obtain what is called the “intertemporal budget constraint”:

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t}$$

In words, the intertemporal budget constraint (“intertemporal” = “across time”) says that the present discounted value of consumption expenditures must equal the present discounted value of income. $\frac{C_{t+1}}{1+r_t}$ is the (real) present value of C_{t+1} . Why is that? The present value is the equivalent amount of consumption I would need today to achieve a given level of consumption in the future. Since saving pays a return of $1 + r_t$, the present value of future consumption would have to satisfy: $(1 + r_t)PV_t = C_{t+1} \Rightarrow PV_t = \frac{C_{t+1}}{1+r_t}$.

Households get utility from consumption. Loosely speaking, you can think about utility as happiness or overall satisfaction. We assume that overall lifetime utility, U , is equal to a weighted sum of utility from consumption in the present and in the future periods:

$$U = u(C_t) + \beta u(C_{t+1}), \quad 0 \leq \beta < 1$$

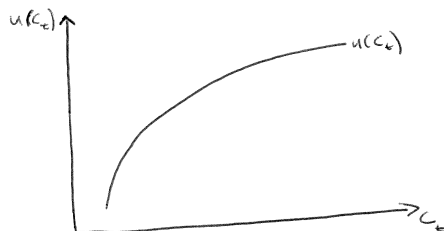
β is what we call the discount factor, and it is constrained to lie within 0 and 1. It is a measure of how the household values current utility relative to future utility. We assume that β must be less than 1, so that the household puts less weight on future utility than the present. This does not seem to be a particularly controversial assumption when looking at how people actually behave in the real world. The bigger is β , the more patient the household is, in the sense that it places a large value on future utility relative to current.

We assume that the utility function mapping consumption into flow utility in each period satisfies the following two properties:

$$u'(C_t) \geq 0$$

$$u''(C_t) \leq 0$$

In words, these properties say that utility is increasing and concave in consumption. Increasing means that more is better – more consumption yields more utility. Concave means that more is better, but at a decreasing rate. This means that the first unit of fruit you consume has higher marginal utility (marginal utility is just the first derivative of the utility function) than the second unit of fruit, which in turn yields more marginal utility than the third unit of fruit, and so on. Below is a plot of what a utility function satisfying these properties might look like:



Some popular utility functions are as follows:

$$u(c_t) = \theta c_t, \quad \theta > 0$$

$$u(c_t) = c_t - \frac{\theta}{2} c_t^2, \quad \theta > 0$$

$$u(c_t) = \ln c_t$$

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma \geq 0$$

The first of these is a linear utility function. Here more is always better, but utility is not strictly concave, so that marginal utility is a constant equal to θ . The second is what is called a quadratic utility function. It is concave (the second derivative is $-\theta$), but it does not always have positive marginal utility; in particular, there is a satiation point at which marginal utility is zero. So long as the parameters are such that the satiation point is not reached, marginal utility is always positive. The third is the log utility function, which satisfies both the properties above. The final utility function is what we sometimes call the iso-elastic utility function (or constant elasticity). σ will have the interpretation as an elasticity, which we will see later. When $\sigma \rightarrow 1$, the isoelastic utility

function converges to the log utility function plus a constant.²

The problem of the household at time t is to choose current and future consumption to maximize lifetime utility, subject to its unified budget constraint. What it is really doing is choosing current consumption and current saving, with saving effectively determining how much consumption it can do in $t + 1$. But because of the way I've written the unified budget constraint, we've eliminated S_t from the analysis. Formally, the problem is:

$$\begin{aligned} \max_{C_t, C_{t+1}} \quad & U = u(C_t) + \beta u(C_{t+1}) \\ \text{s.t.} \quad & \\ & C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t} \end{aligned}$$

As written, this is a constrained, multivariate optimization problem. We will reduce it to an unconstrained, univariate optimization problem by eliminating the constraint. In particular, solve for C_{t+1} from the constraint:

$$C_{t+1} = (1+r_t)(Y_t - C_t) + Y_{t+1}$$

Plug this back into the lifetime utility function, re-writing the maximization problem as just being over C_t :

$$\max_{C_t} \quad U = u(C_t) + \beta u((1+r_t)(Y_t - C_t) + Y_{t+1})$$

To find the optimum, take the derivative with respect to the choice variable, C_t , making use of the chain rule:

$$\frac{dU}{dC_t} = u'(C_t) - \beta u'((1+r_t)(Y_t - C_t) + Y_{t+1})(1+r_t)$$

Now set this equal to zero and simplify, taking note of the fact that $(1+r_t)(Y_t - C_t) + Y_{t+1} = C_{t+1}$ in writing out the first order condition:

$$u'(C_t) = \beta(1+r_t)u'(C_{t+1})$$

This first order condition has the following interpretation: at an optimum, the marginal utility from consuming a little extra today, $u'(C_t)$, must be equal to the marginal utility of saving a little extra today. If you save a little extra today this will leave you with $1+r_t$ extra units of fruit tomorrow, which will yield extra utility of $u'(C_{t+1})(1+r_t)$. The multiplication by β factors in that you discount the future utility payoff. So, in other words, this condition simply says that

²To see this, re-write the isoelastic utility function as $u(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma} = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma}$. In other words, I'm just subtracting a constant, $\frac{1}{1-\sigma}$, from what I showed in the main text. Utility is an ordinal concept, and so we are free to add and subtract constants to them without altering any of the implications of the actual functional form. As $\sigma \rightarrow 1$, we have that $u(c_t) \rightarrow \frac{0}{0}$, so you can use L'Hopital's rule to find the limit, which works out to the natural log.

the household must be indifferent between consuming some more and saving some more at an optimum. If this were not true, the household could increase utility by either consuming or saving some more. We sometimes also refer to this optimality condition as an *Euler equation*: an Euler equation is a dynamic optimality condition, and this is a dynamic (across time) optimality decision for consumption in the present and in the future.

If you've taken intermediate micro, you might recognize this condition as something like a MRS = price ratio condition, where MRS stands for "marginal rate of substitution." The first order condition can be re-written:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t$$

Think about C_{t+1} and C_t as just two different goods. They are different in the time dimension, just as apples and oranges are different in another dimension. $1 + r_t$ is the relative price between first and second period consumption. Consuming an extra fruit today means saving one fewer fruit, which means giving up $1 + r_t$ units of consumption tomorrow. Hence, the price of current consumption relative to the future is $1 + r_t$.

Note that this optimality condition is *not* a "consumption function." A consumption function would express current consumption as a function of income, the interest rate, etc. This condition just relates current to future consumption. It could hold for two low values of current and future consumption, or two higher values of current and future consumption. It is a condition characterizing an optimal consumption allocation. It does not determine current consumption on its own.

We can analyze the consumption problem graphically using an indifference curve - budget line diagram. An indifference curve shows combinations of two goods for which the household achieves the same overall level of utility. Again, just think about consumption of fruit at different points in time as different goods. Suppose we start with an initial consumption bundle, (C_t^0, C_{t+1}^0) . This would yield overall utility:

$$U^0 = u(C_t^0) + \beta u(C_{t+1}^0)$$

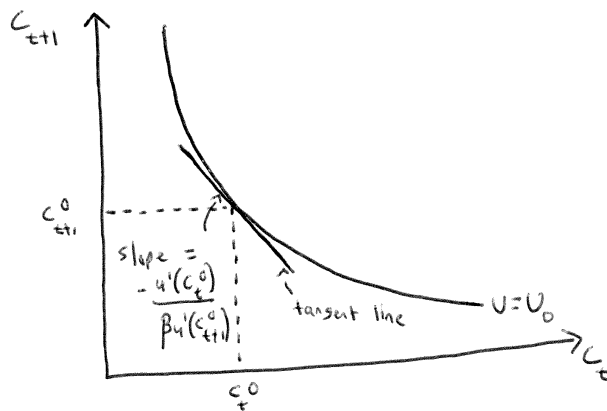
Now we use the concept of the total derivative, which was introduced in the math review notes. The total derivative says that the change in a function is approximately equal to the sum of its partial derivatives evaluated at a point times the change in each of the right hand side variables at a point. Applying that here, we have:

$$dU_t = u'(C_t^0)dC_t + \beta u'(C_{t+1}^0)dC_{t+1}$$

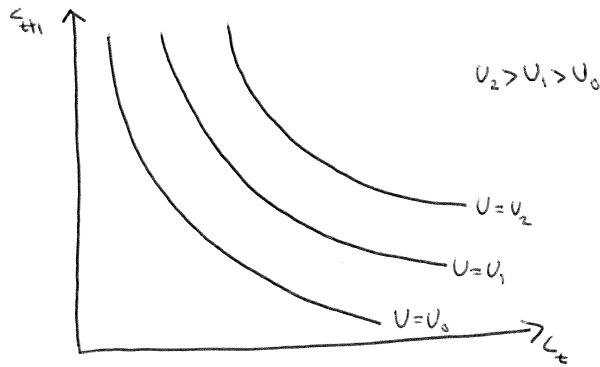
Where $dU_t = U_t - U_t^0$, $dC_t = C_t - C_t^0$, $dC_{t+1} = C_{t+1} - C_{t+1}^0$. Now for utility to stay constant - which is what defines an indifference curve, it must be that $dU_t = 0$. Imposing that and solving, we get:

$$\frac{dC_{t+1}}{dC_t} = -\frac{u'(C_t^0)}{\beta u'(C_{t+1}^0)}$$

This tells us that, in a graph with C_{t+1} on the vertical axis and C_t on the horizontal axis, the slope of the indifference curve at the point (C_t^0, C_{t+1}^0) is equal to the (negative) ratio of marginal utilities evaluated at that point. Assuming that $u''(C_t) < 0$, then this slope will be large when C_t is small (near the origin) and relatively flat when C_t is large (far away from the origin), so an indifference curve should have a “bowed in” kind of shape, looking like the lower southwest part of a circle.



There are different indifference curves associated with different levels of utility. The indifference curve I just described above is the U_0 indifference curve. We could also find a U_1 indifference curve, where $U_1 > U_0$. Since both current and future consumption are “goods,” the “direction of increasing preference” is northeast. That is, indifference curves to the northeast are associated with higher levels of utility than indifference curves more to the southwest.

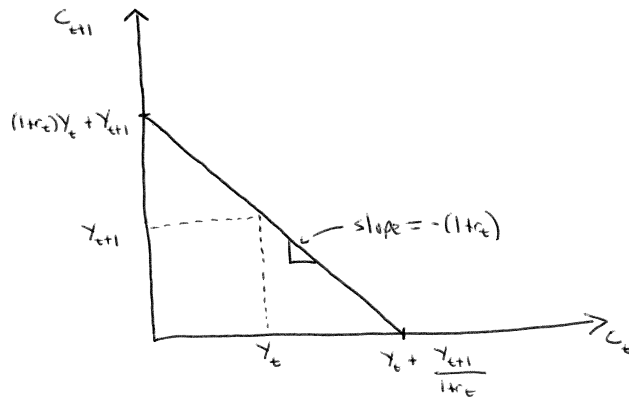


There are some other properties of indifference curves that we will not mention in any depth here. One obvious one is that indifference curves cannot cross. Crossing indifference curves is a logical impossibility when one recognizes that each indifference curve is associated with a particular level of overall utility. If indifference curves crossed, this would mean that one gets different levels of utility at the same consumption bundle, which is impossible. Wikipedia has a pretty good page on indifference curves: http://en.wikipedia.org/wiki/Indifference_curve.

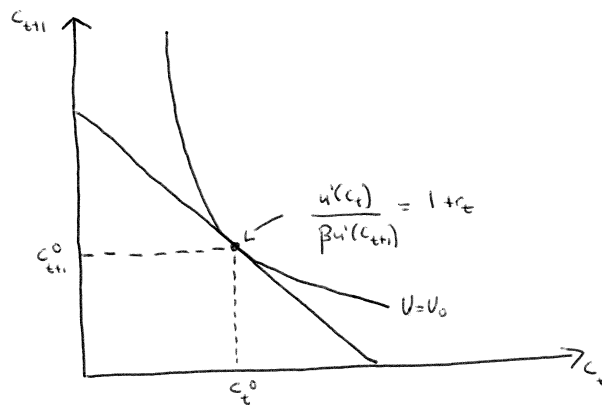
The budget line is a graphical depiction of the intertemporal budget constraint derived above. Solve for C_{t+1} in terms of C_t , the real interest rate, and income levels:

$$C_{t+1} = (1 + r_t)Y_t + Y_{t+1} - (1 + r_t)C_t$$

The budget line intercepts the vertical axis at $C_{t+1} = (1 + r_t)Y_t + Y_{t+1}$ and intersects the horizontal axis at $C_t = Y_t + \frac{Y_{t+1}}{1+r_t}$. It has slope $\frac{dC_{t+1}}{dC_t} = -(1 + r_t)$. Note also that it must cross through the “endowment point” at which $C_t = Y_t$ and $C_{t+1} = Y_{t+1}$. It is always possible to just consume the entirety of income each period, and doing this exhausts resources. It is sometimes helpful to label the endowment point when drawing the budget line.

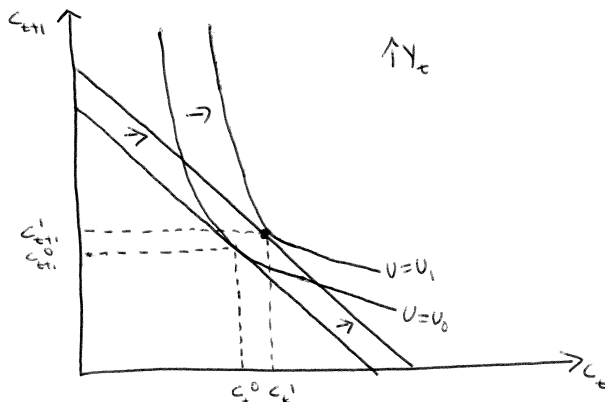


Points inside the budget line are feasible but leave resources unused. Points outside (e.g. to the northeast) of the budget line are not feasible. Graphically, we can think about the household's problem as to pick a point on the highest possible indifference curve that is consistent with the budget constraint. This will occur when the indifference curve just "kisses" the budget constraint. Mathematically, this occurs where the curve is tangent to the budget line, so this optimality condition is sometimes called a "tangency condition." This tangency/optimality condition is shown in the plot below. Tangency means that the slopes are equal. As we can see, the slopes being equal is exactly the mathematical condition characterizing the optimum we derived above.

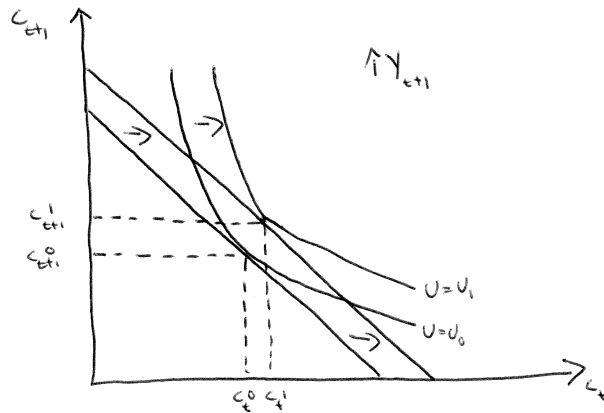


Now that we've graphically characterized the optimal consumption bundle, we can do a couple of exercises. First, suppose that current period income increases, that is that Y_t becomes bigger. We can see in the plot below that this will shift the budget line outward – both the horizontal and vertical axis intercepts increase, with the horizontal intercept increasing by the change in Y_t and

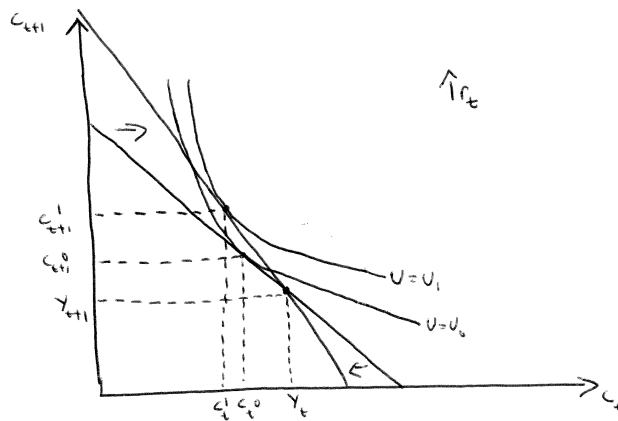
the vertical axis intercept increasing by $1 + r_t$ times the change in Y_t . The slope of the budget line is unchanged. The new point of tangency will be somewhere to the “northeast,” with the household increasing consumption in both periods. We know that consumption in both periods will go up because these are “normal” goods. In the language of intermediate micro, there is a positive income effect that makes the household want to consume more of all the goods in the basket, where current and future consumption are just different goods in the basket, if you will. In contrast, there is no substitution effect because the relative price between current and future consumption, $1 + r_t$, is unchanged. Hence, consumption in both the present and in the future should go up. Consumption in the future going up means that current saving must go up – the household is evidently increasing its current consumption by less than the change in current income if it is going to consume more in the future. Another way of putting this is that the “marginal propensity to consume” out of current income ought to be less than one.



Next, suppose that the household learns that its future income, Y_{t+1} , will increase. Qualitatively, this has a very similar effect to what we observed in the case of an increase in current income. In particular, both the horizontal and vertical axes of the budget line increase – the vertical axis by the change in Y_{t+1} and the horizontal axis by the change in Y_{t+1} divided by $1 + r_t$. The slope of the budget line is unchanged. We can see this in the plot below, which looks very similar to the case above. The outward, parallel shift in the budget line means that the household will want to increase consumption in both periods. This is again because this change is a pure income effect – the household feels richer and there is no change in the relative price of the two goods (consumption today and consumption tomorrow), and so the household increases consumption of both. Increasing current consumption when current income is not changing means that the household must reduce its saving (or increase its borrowing). This is an important result – current consumption depends not only on current income, but also on future income (or, in a world with uncertainty, expectations about future income).



Finally, let's look at what happens when r_t increases. This is going to have the effect of pivoting the budget line through the endowment point, with the budget line becoming steeper. We can see that, holding income in each period fixed, the horizontal axis intercept must shift in. In contrast, the vertical axis intercept must shift up. The budget line must pivot through the endowment point because it is always possible to consume the endowment each period regardless of what r_t is. Below I have shown in the indifference curve-budget line diagram what might happen when r_t increases.



In the paragraph above I said what “might” happen when r_t increases. It turns out that it is theoretically ambiguous what effect an increase in the real interest rate will have on current consumption. The reason why is that, unlike the cases where income changed, there are both income and substitution effects at work here. The substitution effect is always to reduce current consumption and increase future consumption – $1 + r_t$ is the relative price of current consumption, so an increase in r_t makes people want to shift away from current consumption and into future

consumption by saving more. The sign of the income effect depends on whether the household is initially a saver (consumption less than current income) or a borrower (consumption greater than current income). I drew the figure above where the household is initially a saver, with $C_t^0 < Y_t$. If the household is initially a saver, then the income effect is positive – the household feels richer, and will want to consume more in both periods. Intuitively this is clear – the household was already saving, so an increase in the real interest rate means it will get a bigger return on that saving and hence more income in the next period. You can also see this by simply looking at the plot – if the household is initially a saver, then it is guaranteed to be able to get to a higher indifference curve when r_t increases. We can see that by noting that the initial optimality point is strictly inside the new budget line. The higher income the household receives in the second period from its existing saving makes it want to consume more in both periods. Hence, the income effect goes in the opposite direction of the substitution effect for first period consumption – the substitution effect says to consume less, whereas the income effect is to consume more. Which one dominates is unclear – it depends on the nature of preferences and how much the household was initially saving. Now if the household were a borrower, with $C_t^0 > Y_t$, then the income effect would be negative. Intuitively the household would have to pay back more on its borrowing, reducing its future income. This effect would make the household feel poorer, which would lead it to want to reduce both future and current consumption. Hence, if the household is a borrower, the income and substitution effects go in the same direction, leading the household to definitely reduce first period consumption.

I drew the figure above where the household is initially a saver. This means that it is ambiguous as to whether the income or the substitution effect dominates, and hence ambiguous as to whether current period consumption will increase or decrease. I drew the diagram where current period consumption nevertheless decreases – that is, I assumed that the substitution effect dominates. *Unless otherwise noted, we are going to in this class assume that the substitution effect dominates, so that current consumption is decreasing in the real interest rate.* This seems to be the empirically plausible case.

2.1 The Consumption Function

As emphasized above, the tangency condition or Euler equation is *not* a consumption function. The Euler equation is a condition between current and future consumption (and the relative price between the two, the real interest rate). A consumption function expresses current consumption as a function of things other than future consumption: income, the interest rate, and parameters.

From the indifference curve exercises above, we see that consumption evidently depends on current income, future income, and the real interest rate. Generally:

$$C_t = C(Y_t, Y_{t+1}, r_t)$$

Here $C(\cdot)$ is a function mapping income and the interest rate into consumption. We know something about its partial derivatives. In particular, $\frac{\partial C_t}{\partial Y_t} > 0$ and $\frac{\partial C_t}{\partial Y_{t+1}} > 0$. That is, consumption is

increasing in both current and future income. As we noted above, the sign of $\frac{\partial C_t}{\partial r_t} > 0$ is theoretically ambiguous. *Unless otherwise indicated*, we will assume that the substitution effect dominates, so that consumption is decreasing in the real interest rate. This seems to be the empirically relevant case, and it also works out that way with most of the functional forms that we will use.

For a general specification of utility, to derive the consumption function you would combine the first order/tangency condition with the budget constraint. The first order equation has two unknowns: C_t and C_{t+1} . The budget constraint also has two unknowns: C_t and C_{t+1} . Combining them yields two equations in two unknowns, and we can solve for the values of current and future consumption that make both expressions hold with equality. Doing this will allow us to back out an analytic expression for the consumption function.

Let's look at this in action. In particular, assume that the within-period utility function is log, so that $u(C_t) = \ln C_t$. The first order condition can be written:

$$\frac{C_{t+1}}{C_t} = \beta(1 + r_t)$$

“Solving” for C_{t+1} , we get:

$$C_{t+1} = \beta(1 + r_t)C_t$$

Now take this and plug it into the intertemporal budget constraint:

$$C_t + \frac{\beta(1 + r_t)C_t}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t}$$

Now simplify:

$$\begin{aligned} C_t + \beta C_t &= Y_t + \frac{Y_{t+1}}{1 + r_t} \\ (1 + \beta)C_t &= Y_t + \frac{Y_{t+1}}{1 + r_t} \\ C_t &= \frac{1}{1 + \beta}Y_t + \frac{1}{(1 + \beta)(1 + r_t)}Y_{t+1} \end{aligned}$$

The final line gives us the consumption function – how current consumption depends on current income, future income, and the real interest rate. We can calculate partial derivatives as follows:

$$\begin{aligned} \frac{\partial C_t}{\partial Y_t} &= \frac{1}{1 + \beta} > 0 \\ \frac{\partial C_t}{\partial Y_{t+1}} &= \frac{1}{(1 + \beta)(1 + r_t)} > 0 \\ \frac{\partial C_t}{\partial r_t} &= -\frac{1}{(1 + \beta)(1 + r_t)^2}Y_{t+1} \end{aligned}$$

There are a couple of interesting things to note here. First, the partial with respect to current

income, or what is sometimes called the “marginal propensity to consume,” is positive, and bound between $\frac{1}{2}$ ($\beta \rightarrow 1$) and 1 ($\beta \rightarrow 0$). In other words, the bigger is β , so the more patient is the household, the lower is the marginal propensity to consume. Second, the partial with respect to future income is also positive. This being positive relies on $r_t > -1$. Note that the real interest rate *can be negative*. Remember, all the real interest rate says is how many goods you get back tomorrow for giving up a good today. You may be willing to take a deal in which this return is negative, so that you get back fewer goods tomorrow than you give up today. This is driven by the assumption (noted above), that fruit is not directly storable – if you could store your fruit, one fruit today left in the pantry would be one fruit tomorrow still in the pantry, and no household would accept a negative real interest rate, because the best outside option would be holding on to it and earning zero return. But if storability is not an option, the household may be better off taking a negative real interest rate than not saving at all. Nevertheless, r_t must be > -1 . If the real interest rate were -1 , this would mean that you give up a fruit today for nothing in return. You would never do that – you’d be better off eating the fruit today and having nothing tomorrow. Since $r_t > -1$, the partial with respect to future income must be positive. Whether the marginal propensity to consume out of future income is bigger or smaller than current income depends on the sign of r_t . If $r_t > 0$, the partial with respect to future income is smaller than with respect to current income. If $r_t < 0$, the reverse is true – a one fruit increase in future income would have a bigger effect on current consumption than would a one fruit increase in current income. This is perhaps easiest to see by noting that $\frac{Y_{t+1}}{1+r_t}$ is the present value of future income. Looking at the derivatives, $\frac{1}{1+\beta}$ is the partial with respect to the present value of income in either period (the present value of current income is just current income). If $r_t > 0$, the present value of future income is smaller than current income, and so the overall MPC is smaller. If $r_t < 0$, then the reverse is true.

Finally, let’s look at the derivative with respect to the real interest rate. Here we see that, as long as $Y_{t+1} > 0$, then the derivative must be negative. That is, consumption is decreasing in the real interest rate, unless the household was going to have no income in the future. The case of no income in the future would correspond to the case of the income and substitution effects perfectly canceling out that we discussed above in analyzing indifference curves. This seems reasonable to assume that people have at least some income in the future, and hence, with log utility, we see that current consumption is in fact decreasing in the real interest rate.

3 Some Extensions of the Two Period Model

In this section we are going to look at three different implications of the two period model. These are (i) “permanent” versus “transitory” changes in income, with an application to tax cuts; (ii) the role of wealth; (iii) and the role of uncertainty.

3.1 “Permanent” vs. “Transitory” Income

For a generic specification of utility, the consumption function can be characterized as:

$$C_t = C(Y_t, Y_{t+1}, r_t)$$

As shown above, the partial derivatives of the consumption function with respect to both current and future income are positive. Given this, it should come as no surprise that consumption ought to react more to changes in income the more persistent the change in income is. Suppose that we have a simultaneous increase in both current and future income. Using the total derivative, the approximate change in consumption would be the sum of the partial derivatives (evaluated at a point) times the changes in income:

$$dC_t \approx \frac{\partial C}{\partial Y_t} dY_t + \frac{\partial C}{\partial Y_{t+1}} dY_{t+1}$$

To see this most clearly, suppose that there is a “permanent” change in income, so that current and future income increase by exactly the same amount. Call this common amount $d\bar{Y}$. Simplifying, we see that the approximate change in consumption for a one unit “permanent” increase in income is equal to the sum of the partials:

$$\frac{dC_t}{d\bar{Y}} \approx \frac{\partial C}{\partial Y_t} + \frac{\partial C}{\partial Y_{t+1}}$$

As we discussed in the previous section, both of these partial derivatives should be positive. This means that a change in income that persists into both periods ought to have a much larger effect on consumption than a change that only lasts one period. To use a concrete example, we derived a consumption function for the case of log utility in the last section. If we approximate $r_t \approx 0$ and $\beta \approx 1$, then the partial derivative of the consumption function with respect to income in either period is $\frac{1}{2}$. Continuing with that approximation, we can deduce that a “transitory” change in income in just the first period ought to be half consumed and half saved. In contrast, a “permanent” change in income should result in a roughly one-for-one reaction of consumption. Another way to phrase this is as follows: the marginal propensity to consume out of income will depend on how persistent the change in income is. The more persistent the change in income, the more consumption ought to react.

This insight has potentially important implications for policy. During the last several recessions there have been “stimulus” packages which, among other things, cut taxes for most households. We can fairly easily modify the household problem to include taxes. Let T_t and T_{t+1} be tax payments in periods t and $t + 1$, respectively. Since income is exogenously given in this exercise, just think of taxes as the government taking away some of the household’s endowment of fruit. The period-by-period budget constraints look like:

$$C_t + S_t = Y_t - T_t$$

$$C_{t+1} = Y_{t+1} - T_{t+1} + (1 + r_t)S_t$$

These can be combined into one intertemporal budget constraint, which has the same interpretation as earlier, just with “net income” on the right hand side:

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t - T_t + \frac{Y_{t+1} - T_{t+1}}{1 + r_t}$$

The consumption function then takes the same generic form as above, but with net income as the arguments:

$$C_t = C(Y_t - T_t, Y_{t+1} - T_{t+1}, r_t)$$

Changes in taxes work just like changes in income. A change in current taxes will lead to some increase in consumption (maybe around half a unit increase in C_t for every one unit reduction in T_t), but a significant fraction of the tax cut will end up being saved. In contrast, a “permanent” tax cut – a policy which lowers T_t and T_{t+1} – will lead to a much larger reaction of consumption. Much of the policy discussion has been aimed at giving people more net income with the hope that doing so will stimulate consumption, which will in turn lead to short run job creation. This discussion indicates that how much consumption gets stimulated will depend upon on how persistent the change in taxes is.

3.2 Wealth

In the previous sections we assumed that the household begins life with no stock of wealth – its only source of resources is its income. We can fairly easily modify the household problem to include wealth.

Let B_0 denote the stock of assets with which the household enters life. Its utility is the same. The two within-period budget constraints can be written:

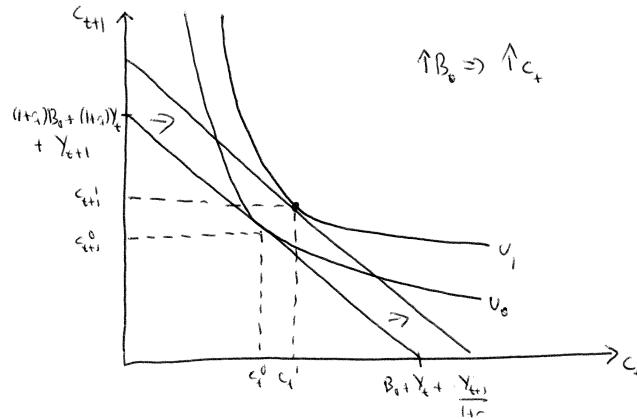
$$C_t + S_t = B_0 + Y_t$$

$$C_{t+1} = Y_{t+1} + (1 + r_t)S_t$$

Only the first period constraint is affected. Initial wealth basically functions as extra income in the first period. If the household chooses not to spend its wealth, it goes into saving, with $S_t = Y_t + B_0 - C_t$, and that savings earns interest r_t between periods. Combining the two within-period constraints gives rise to a modified intertemporal budget constraint:

$$C_t + \frac{C_{t+1}}{1+r_t} = B_0 + Y_t + \frac{Y_{t+1}}{1+r_t}$$

This looks very much like the original intertemporal budget constraint, and as I alluded to above, we can really just think about B_0 as being another source of first period income. Changes in B_0 will function just like changes in first period income – they will shift the budget constraint in or out, leading to changes in consumption and saving:



Although it may seem somewhat trivial since you can really think about wealth as first period income here, I think it is worth pointing the wealth channel out directly. In the last fifteen years there have been episodes where “wealth” seems to have played a role in actual consumption patterns. The two most important sources of wealth for most households are (i) housing and (ii) stock market holdings. In the mid-1990s, stock prices rose. This had the effect of raising wealth for households, which made them feel richer and led them to spend more, helping to fuel a boom. In the mid-2000s home prices soared, leading to increasing consumption for similar reasons. In late 2006 and early 2007, however, home prices came crashing down. This had a negative wealth effect that made households want to reduce their consumption, and was an important contributor in the recent recession.

3.3 Uncertainty

Up until this point we have assumed that future income is known in the present – that is, at time t households know Y_{t+1} with certainty. In this subsection we consider the implications of how uncertainty impacts consumption decisions.

Suppose that future income can take on two values: $Y_{t+1}^h > Y_{t+1}^l$. Let the probability of the “high state” occurring be p , with the probability of the “low state” of $1 - p$. Then the expected value of Y_{t+1} is equal to:

$$E(Y_{t+1}) = pY_{t+1}^h + (1 - p)Y_{t+1}^l$$

The optimization problem of the household has to be re-cast slightly. In particular, the household will want to maximize *expected* utility subject to the intertemporal budget constraint. It is expected utility because, if future income is not known, then future consumption cannot be known with certainty. If income ends up high, consumption in the future will be relatively high. But if income ends up low, then consumption will be low. Not knowing future consumption with certainty means that one cannot know future utility with certainty. Hence, when making consumption/saving decisions in the present, one seeks to maximize expected utility. The first order condition ends up looking the same as in the case with certainty, but with an expectation operator. The optimality condition, or Euler equation, is:

$$u'(C_t) = \beta(1 + r_t)E(u'(C_{t+1}))$$

Note that what shows up on the right hand side is the expected marginal utility of consumption, which is not the same thing as marginal utility of expected consumption. With two possible realizations of future income, there are two possible realizations of future consumption, given current consumption, current income, and the real interest rate:

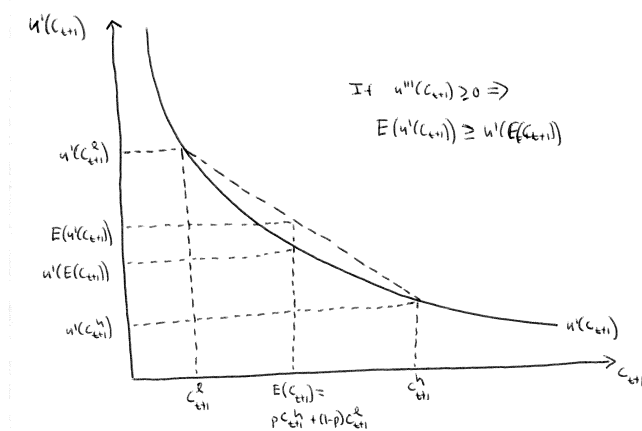
$$\begin{aligned} C_{t+1}^h &= Y_{t+1}^h + (1 + r_t)(Y_t - C_t) \\ C_{t+1}^l &= Y_{t+1}^l + (1 + r_t)(Y_t - C_t) \end{aligned}$$

The expected value of consumption in the second period is $E(C_{t+1}) = pC_{t+1}^h + (1 - p)C_{t+1}^l$. The key insight to understanding how uncertainty impacts consumption is that *expected marginal utility is not, in general, the same thing as marginal utility evaluated at the expected value of future consumption*. Only in the special case in which marginal utility is linear, which occurs with the quadratic utility function shown above, will expected marginal utility coincide with the marginal utility of expected consumption.

Let's suppose that the third derivative of the consumption function is positive: $u'''(C_t) > 0$. The third derivative of the utility function is the second derivative of marginal utility. Hence, the third derivative is a measure of the curvature of marginal utility. If the third derivative is positive, then it means that marginal utility is decreasing, but it flattens out as C_t gets big. In words, that means that marginal utility is decreasing in C_t , but it flattens out as C_t gets big (i.e. the slope of marginal utility becomes less negative). The log utility function has this property, for example. When $u(C_t) = \ln C_t$, then $u'(C_t) = C_t^{-1} > 0$, $u''(C_t) = -C_t^{-2} < 0$, and $u'''(C_t) = 2C_t^{-3} > 0$.

Below is a plot of marginal utility of future consumption, $u'(C_{t+1})$, as a function of C_{t+1} . Graphically, we can see the third derivative being positive means that marginal utility is “bowed in” (the plot of marginal utility actually looks like an indifference curve, or the lower southwest portion of a circle). Expected consumption is equal to $pC_{t+1}^h + (1 - p)C_{t+1}^l$. Expected marginal

utility is $pu'(C_{t+1}^h) + (1-p)u'(C_{t+1}^l)$. Graphically, we can calculate expected marginal utility by drawing a straight line between C_{t+1}^h and C_{t+1}^l , and evaluating the value of the straight line at the expected value of consumption. For a proof of this, see the discussion immediately following the plot. Because of the “bowed in” shape of marginal utility, the straight line between the points lies everywhere above the marginal utility curve. Put differently, expected marginal utility must be greater than marginal utility evaluated at expected future consumption. Mathematically, $E u'(C_{t+1}) > u'(E(C_{t+1}))$. This is a statement of Jensen’s inequality, which says that the expected value of a convex function is greater than the function of the expected value. If $u'''(C_t) > 0$, then $u'(C_{t+1})$ is a convex function, because its second derivative (the third derivative of the utility function) is positive.



(You can feel free to skip this paragraph if you are willing to take the above paragraph at face value). We can prove that expected marginal utility must equal the point on the line which connects marginal utility at the high and low values evaluated at the mean value of consumption using a little bit of algebra. The slope of the line connecting those two points is simply “rise over run”:

$$\text{slope} = \frac{u'(C_{t+1}^h) - u'(C_{t+1}^l)}{C_{t+1}^h - C_{t+1}^l}$$

Let’s treat the value of the line evaluated at $E(C_{t+1})$ as an unknown; for simplicity call it x . The slope at x is just equal to rise over run as well:

$$\text{slope} = \frac{x - u'(C_{t+1}^l)}{E(C_{t+1}) - C_{t+1}^l}$$

Because we have drawn a line, the slope must be the same at all points. This means that these two expressions for slope must be equal. Hence, we can set them equal and solve for x :

$$\frac{x - u'(C_{t+1}^l)}{E(C_{t+1}) - C_{t+1}^l} = \frac{u'(C_{t+1}^h) - u'(C_{t+1}^l)}{C_{t+1}^h - C_{t+1}^l}$$

First, plug in for the expected value of consumption, which can be written $E(C_{t+1}) = p(C_{t+1}^h - C_{t+1}^l) + C_{t+1}^l$ and then start simplifying:

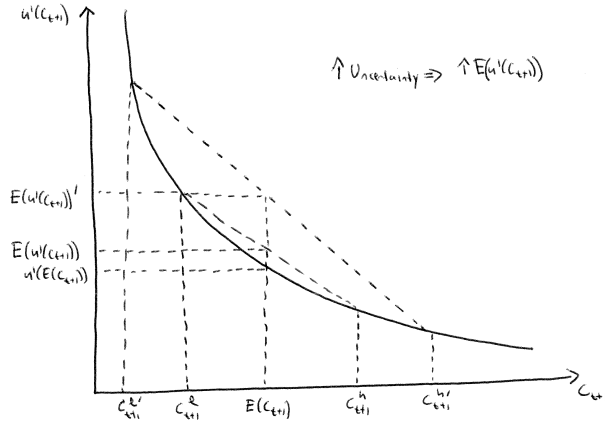
$$\begin{aligned} \frac{x - u'(C_{t+1}^l)}{p(C_{t+1}^h - C_{t+1}^l)} &= \frac{u'(C_{t+1}^h) - u'(C_{t+1}^l)}{C_{t+1}^h - C_{t+1}^l} \\ x - u'(C_{t+1}^l) &= \frac{(p(C_{t+1}^h - C_{t+1}^l))(u'(C_{t+1}^h) - u'(C_{t+1}^l))}{C_{t+1}^h - C_{t+1}^l} \\ x &= \frac{(p(C_{t+1}^h - C_{t+1}^l))(u'(C_{t+1}^h) - u'(C_{t+1}^l))}{C_{t+1}^h - C_{t+1}^l} + u'(C_{t+1}^l) \\ x &= \frac{(p(C_{t+1}^h - C_{t+1}^l))(u'(C_{t+1}^h) - u'(C_{t+1}^l)) + (C_{t+1}^h - C_{t+1}^l)u'(C_{t+1}^l)}{C_{t+1}^h - C_{t+1}^l} \\ x &= \frac{p(C_{t+1}^h - C_{t+1}^l)u'(C_{t+1}^h) + (1-p)(C_{t+1}^h - C_{t+1}^l)u'(C_{t+1}^l)}{C_{t+1}^h - C_{t+1}^l} \\ x &= pu'(C_{t+1}^h) + (1-p)u'(C_{t+1}^l) = Eu'(C_{t+1}) \end{aligned}$$

This proves that the line connecting marginal utilities at the two possible consumption values evaluated at the mean consumption value is equal to expected marginal utility. And since the the line lies above the curve, expected marginal utility is greater than marginal utility of expected consumption.

If there is uncertainty over future income, and the third derivative of utility is positive, then marginal utility of future consumption will be higher than if there were no uncertainty. Re-written slightly, the Euler equation says:

$$\frac{u'(C_t)}{\beta(1+r_t)} = E(u'(C_{t+1}))$$

Looking at this first order condition, we can qualitatively think about what happens when there is an increase in uncertainty. An increase in uncertainty that leaves the expected value (mean) of future income unaffected must raise expected marginal utility. We can see this depicted graphically below. The right hand side in the above first order condition thus increases. To make the optimality condition hold, then the left hand side must also get bigger. For a given interest rate, this means that the current marginal utility of consumption must increase. Marginal utility of current consumption getting bigger requires current consumption to *fall*. Put differently, households will react to higher uncertainty by trying to reduce current current consumption, or equivalently by trying to increase current saving. We call this *precautionary saving*.



The intuition for precautionary saving can be seen from the graph above. If marginal utility is convex (i.e. if the third derivative of the utility function is positive), and uncertainty increases then the “pain” from the bad state being realized is worse than the “gain” from the good state being realized. Households will react to this by trying to “save for the rainy day” – by building up a larger stock of savings, the household will be better able to minimize its utility losses from the bad state occurring. The end result is the following: an increase in uncertainty over future income, holding all other factors constant, will lead the household to try to reduce its current consumption.

4 Multi-Period Generalization and the Life Cycle

All of the basic insights of the two period model carry over to a multi-period extension of the model. The distinction between “permanent” and “transitory” income becomes even stronger in a multi-period setting, however, and we can also think about how consumption and income ought to vary over the “life cycle” when there are multiple periods. For this section we revert to assuming that future income is known with certainty.

Instead of assuming that households just live for two periods (t and $t + 1$), let’s assume that they live for many periods: $t, t + 1, t + 2, \dots, t + T$ ($T + 1$ total periods). Lifetime utility is now:

$$U = u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}) + \beta^3 u(C_{t+3}) + \dots + \beta^T u(C_{t+T})$$

Here each period into the future gets multiplied again by β . β is still the relative weight placed on future consumption between any two adjacent periods. Between today and several periods into the future, however, the weight on future utility is β raised to the number of periods in between. For example, if $T = 50$ and $\beta = 0.95$, $\beta^{50} = 0.08$, so that the household places relatively little weight (0.08) on utility flows 50 periods from now relative to the present. We can equivalently write lifetime utility using the summation operator:

$$U = \sum_{j=0}^T \beta^j u(C_{t+j})$$

As in the two period model, the household faces a sequence of budget constraints: one for each period of time:

$$\begin{aligned} C_t + S_t &= Y_t \\ C_{t+1} + S_{t+1} &= Y_{t+1} + (1 + r_t)S_t \\ C_{t+2} + S_{t+2} &= Y_{t+2} + (1 + r_{t+1})S_{t+1} \\ C_{t+3} + S_{t+3} &= Y_{t+3} + (1 + r_{t+2})S_{t+2} \\ &\vdots \\ C_{t+T} &= Y_{t+T} + (1 + r_{t+T-1})S_{t+T-1} \end{aligned}$$

At this point, it is important to make the distinction between saving and savings. Saving (the term we've been using) is a flow variable. Savings is a stock, with saving the change (either positive or negative) to the stock between periods. In the two period model, we didn't need to make any distinction – when you begin life with no savings, and end life with no savings, the stock and the flow end up being the same thing. Here that is not the case. Above the S s are *stocks*: S_t is the stock of savings the household takes into period $t + 1$, S_{t+1} is the stock of savings the household takes into period $t + 2$, and so on. The flow – saving with an s – is the change in the stocks. For example, saving in period $t + 1$ is $S_{t+1} - S_t$: the change in the stock. In period t stock and flow are one in the same, given the assumption of no starting stock of savings. Also, a note on the timing: r_{t+j} is the real interest rate at time $t + j$ that pays off in the next period, $t + j + 1$.

As in the two period model, it is helpful to collapse these period budget constraints into one unified, intertemporal budget constraint. To do so, it is helpful to assume that the interest rate is constant across time; that is, that $r_{t+j} = r$ for $j = 0, 1, \dots, T - 1$. To collapse the budget constraints into one, let's start in the final period and work our way backwards. In the final period, we see that stock of savings inherited must satisfy:

$$S_{t+T-1} = \frac{C_{t+T}}{1+r} - \frac{Y_{t+T}}{1+r}$$

Now go to the previous period and plug this in:

$$\begin{aligned} C_{t+T-1} + S_{t+T-1} &= Y_{t+T-1} + (1+r)S_{t+T-2} \\ C_{t+T-1} + \frac{C_{t+T}}{1+r} &= Y_{t+T-1} + \frac{Y_{t+T}}{1+r} + (1+r)S_{t+T-2} \\ S_{t+T-2} &= \frac{C_{t+T-1}}{1+r} + \frac{C_{t+T}}{(1+r)^2} - \frac{Y_{t+T-1}}{1+r} - \frac{Y_{t+T}}{(1+r)^2} \end{aligned}$$

Now go two periods before the final period and do the same:

$$\begin{aligned}
C_{t+T-2} + S_{t+T-2} &= Y_{t+T-2} + (1+r)S_{t+T-3} \\
C_{t+T-2} + \frac{C_{t+T-1}}{1+r} + \frac{C_{t+T}}{(1+r)^2} - \frac{Y_{t+T-1}}{1+r} - \frac{Y_{t+T}}{(1+r)^2} &= Y_{t+T-2} + (1+r)S_{t+T-3} \\
S_{t+T-3} &= \frac{C_{t+T-2}}{1+r} + \frac{C_{t+T-1}}{(1+r)^2} + \frac{C_{t+T}}{(1+r)^2} - \frac{Y_{t+T-2}}{1+r} - \frac{Y_{t+T-1}}{(1+r)^2} - \frac{Y_{t+T}}{(1+r)^3}
\end{aligned}$$

If you are paying attention you'll see that a pattern is developing. If you keep doing this and go back to period t , you end up with:

$$C_t + \frac{C_{t+1}}{1+r} + \frac{C_{t+2}}{(1+r)^2} + \dots + \frac{C_{t+T}}{(1+r)^T} = Y_t + \frac{Y_{t+1}}{1+r} + \frac{Y_{t+2}}{(1+r)^2} + \dots + \frac{Y_{t+T}}{(1+r)^T}$$

We can equivalently write this using summation operator notation:

$$\sum_{j=0}^T \frac{C_{t+j}}{(1+r)^j} = \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j}$$

In words, this just says that the present discounted value of consumption must equal the present discounted value of income. This is nothing more than a multi-period extension of the two period intertemporal budget constraint.

Having collapsed the period-by-period budget constraints into one, we can write the household's problem as choosing, at date t , the entire sequence of lifetime consumption subject to the intertemporal budget constraint:

$$\begin{aligned}
\max_{C_t, C_{t+1}, \dots, C_{t+T}} \quad & U = \sum_{j=0}^T \beta^j u(C_{t+j}) \\
\text{s.t.} \quad &
\end{aligned}$$

$$\sum_{j=0}^T \frac{C_{t+j}}{(1+r)^j} = \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j}$$

It is worth noting that in reality the household is not literally choosing consumption at all future dates; but because savings links budget constraints across time, current saving decisions affect future consumption possibilities, and hence by choosing saving it is as if the household is choosing future consumption. One could proceed exactly as we did in the two period case: solve the constraint for consumption in the last period of life, plug it back into the objective function, and transform the problem into an unconstrained one. This works but it ends up being tedious, because you have to take T derivatives after doing that. The optimality conditions end up working out nicely, though. These are:

$$\begin{aligned}
u'(C_t) &= \beta(1+r)u'(C_{t+1}) \\
u'(C_{t+1}) &= \beta(1+r)u'(C_{t+2}) \\
&\vdots \\
u'(C_{t+T-1}) &= \beta(1+r)u'(C_{t+T})
\end{aligned}$$

A general representation is as follows:

$$u'(C_{t+j}) = \beta(1+r)u'(C_{t+j+1}), \quad j = 0, 1, \dots, T-1$$

Basically, one way to interpret this is that there's an indifference curve - budget line diagram for every two adjacent periods. The optimality condition between the two periods takes the same form as in the two period case, and it still has the same economic interpretation that we offered up in the two period model. In particular, at an optimum the household must be indifferent between consuming a little bit more in period $t+j$ (benefit of $u'(C_{t+j})$) or saving a little bit more (benefit of $\beta(1+r)u'(C_{t+j+1})$). Basically, we can think about the multi-period problem as just being a sequence of adjacent two period problems.

To go much further than this – in particular to derive a consumption function – we need to make some simplifying assumptions. In addition to assuming that r is constant across time, let's assume that $\beta(1+r) = 1$. Looking at the tangency conditions above, if this is true, then it must be the case that the marginal utilities of consumption in any two adjacent periods equal each other: $u'(C_{t+j}) = u'(C_{t+j+1})$. But if the marginal utilities are equal, then it must be the case that the levels of consumption are equal: $C_{t+j} = C_{t+j+1}$. Since this Euler equation must hold for any two adjacent periods in time, this evidently means that consumption must be the same in any period. Let's call this value $C_{t+j} = \bar{C}$, $j = 0, 1, \dots, T$.³

To derive a consumption function for this multi-period model, let's plug this modified Euler equation back into the intertemporal budget constraint. This works out nicely – if consumption in every period is the same, then we can just pull it out of the sum on the left hand side:

$$\begin{aligned}
\sum_{j=0}^T \frac{C_{t+j}}{(1+r)^j} &= \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j} \Rightarrow \\
\bar{C} \sum_{j=0}^T \left(\frac{1}{1+r} \right)^j &= \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j}
\end{aligned}$$

Now, we can actually simplify the summation on the right hand side, since it is just over a constant.

³Looking back to the previous section on uncertainty, I am obviously here ignoring uncertainty. If future income were uncertain, then future marginal utility would be high. This would lead households to consume less today, building up a stock of savings. In expectation, consumption would be expected to grow across time, because the initial savings built up would eventually get spent in expectation.

To see how, let α denote some generic constant with value less than 1. Here I'm letting α play the role of $\frac{1}{1+r}$.⁴ We can write out a generic sum as:

$$S = \sum_{j=0}^T \alpha^j = 1 + \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^T$$

Multiply both sides by α and simplify, solving for S :

$$\begin{aligned} S\alpha &= \alpha + \alpha^2 + \alpha^3 + \dots + \alpha^T + \alpha^{T+1} \\ S - S\alpha &= 1 - \alpha^{T+1} \\ S &= \frac{1 - \alpha^{T+1}}{1 - \alpha} \approx \frac{1}{1 - \alpha} \end{aligned}$$

The last line, with the approximation, follows from the fact that, if T is sufficiently big, then $\alpha^{T+1} \approx 0$. As noted above, let $\alpha = \frac{1}{1+r}$. Suppose that $r = 0.05$ and $T = 50$; that is, a five percent interest rate and a 50 year horizon. Using those numbers, we'd get $\left(\frac{1}{1+r}\right)^{T+1} = 0.08$, which is pretty small. Using this approximate formula, we can simplify the sum above:

$$\sum_{j=0}^T \left(\frac{1}{1+r}\right)^j = \frac{1}{1 - \frac{1}{1+r}} = \frac{1+r}{r}$$

We can take this expression, along with the fact that consumption will be constant under our assumptions, to the intertemporal budget constraint. We get:

$$\bar{C} \frac{1+r}{r} = \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j}$$

Simplifying:

$$\bar{C} = \frac{r}{1+r} \sum_{j=0}^T \frac{Y_{t+j}}{(1+r)^j}$$

In words, this expression says that you will set consumption in each period of life equal to $\frac{r}{1+r}$ times the present discounted value of the stream of income. Note that $\frac{r}{1+r}$ should be a relatively small number; e.g. if $r = 0.05$, then $\frac{r}{1+r} = 0.0475$.

We can use this expression for the consumption function to calculate some partial derivatives to figure out how consumption will react to changes in income. First, let's look at the effect of a change in current income, where the current period is taken to be the first period, period t . The partial derivative is:

⁴Note that r_t can be negative at any point in time as discussed above, but here I am assuming that $\beta(1+r) = 1$. Since $\beta < 1$, $r > 0$ by assumption here. This means that $\frac{1}{1+r} < 1$, which means the sum "converges" in the sense that $\left(\frac{1}{1+r}\right)^T \rightarrow 0$ as $T \rightarrow \infty$.

$$\frac{\partial \bar{C}}{\partial Y_t} = \frac{r}{1+r}$$

This says that, if current income increases by 1 unit (remember everything here is real), then consumption will increase by $\frac{r}{1+r}$ units, which is, as we saw above, a small number. Since consumption must be the same in every period of life, not only does current consumption increase, but consumption in every period must go up by the same amount. Suppose that $r = 0.05$. When current income goes up by 1, consumption in each period should evidently go up by 0.0475. This means that saving in the first period would go up by 0.9525, or $1 - 0.0475$. The increase in saving in the first period would finance the higher consumption in future periods. The bottom line here is that if current income goes up, the household should increase its consumption only by a little – most of the increase in income goes into higher saving. The higher saving allows consumption in all future periods of life to also increase by a little. Put differently, in a multi-period forward-looking model, the marginal propensity to consume out of current income ought to be small. This is really just a generalization of what we saw in the two period model. In that framework the household would respond to a change in current income by increasing consumption in both periods. Increasing consumption in both periods requires increasing current consumption by less than the change in current income – saving must go up, with the increase in current saving financing the change in future consumption. For the two period model with, for example, log utility, I argued that a one unit increase in current income would be allocated roughly 1/2 to current consumption and 1/2 to saving. With more than two future periods, this allocation evidently switches over much more strongly into saving. The household will want to increase consumption in all subsequent periods of life – i.e. it will want to “smooth” its consumption. If there are many periods this requires increasing current consumption only by a little bit, and hence current saving by a lot.

We can also take the partial derivative with respect to income in any future period of life, $t + j$:

$$\frac{\partial \bar{C}}{\partial Y_{t+j}} = \frac{r}{1+r} \left(\frac{1}{1+r} \right)^j$$

The partial with respect to current income shown above is a special case of this when $j = 0$. Since for this exercise $r > 0$, when j is bigger, the partial is smaller. This has the following interpretation – the farther out in the future is the increase in income, the less consumption reacts. It is again important to note that consumption will react in all periods, not just the period of the change in income. The fact that current consumption reacts to a change in future income means that current saving must fall when future income is expected to increase.

Finally, let’s consider the following thought experiment. Instead of supposing that income increases in one period only, what would happen if income increased by the same amount in all periods? Think about getting a raise – your salary will be higher today but also in the future. This is what we might call a change in “permanent” income. In contrast, a one period change in income is transitory – for example, you find 20 bucks on the street or you when you win a bet. These are not changes in income that should persist into the future. This is again just the extension of the

“permanent” change in income that we saw in the two period model.

Let’s again use the concept of the total derivative. The total derivative says that the change in the left hand side variable is equal to the sum of the partial derivatives evaluated at a point times the changes in the right hand side variables. Since the consumption function here is linear in current and future income, the total derivative here will actually hold with equality – it is not just an approximation. The total derivative is:

$$d\bar{C} = \frac{r}{1+r}dY_t + \frac{r}{1+r}\frac{1}{1+r}dY_{t+1} + \frac{r}{1+r}\left(\frac{1}{1+r}\right)^2 dY_{t+2} + \dots + \frac{r}{1+r}\left(\frac{1}{1+r}\right)^T dY_{t+T}$$

Using summation notation, this simplifies to:

$$d\bar{C} = \frac{r}{1+r} \sum_{j=0}^T \left(\frac{1}{1+r}\right)^j dY_{t+j}$$

Now, assume that income increases in each period by the same amount, so that $dY_{t+j} = d\bar{Y}$. If these are all the same, we can take them out of the sum, leaving:

$$d\bar{C} = d\bar{Y} \frac{r}{1+r} \sum_{j=0}^T \left(\frac{1}{1+r}\right)^j$$

Something pretty cool happens – the sum is the inverse of $\frac{r}{1+r}$, so these cancel and we are left with:

$$\frac{d\bar{C}}{d\bar{Y}} = 1$$

In words, what this says is that if income increases by 1 unit in every period, then consumption will also increase by 1 in every period. Put differently, the marginal propensity to consume out of “permanent income” is 1. By permanent income I mean a change in income that lasts in all periods of life, not just one. The so-called “permanent income hypothesis,” credited to Friedman (1957), suggests that consumption ought to be set on the basis of permanent income, with changes in transitory income having little effect on consumption.

Compare the derivative of consumption with respect to permanent income to the derivative of consumption with respect to current income. If current income goes by 1 and it is only expected to be higher for one period, there is a very small effect on current consumption. In contrast, if current *and all future period* income goes up by 1, then consumption reacts one for one. This has a very important implication – the amount by which consumption responds to income depends on how persistent the income change is. You can see that this result is just an extension of what we saw in the two period model. We’ve looked at two polar cases: if the change in income is very transitory (e.g. just lasts one period), then the change in consumption should be very small; in contrast, if the change in income is very persistent (e.g. permanent), then consumption should react one-for-one. For in-between levels of persistence, you’ll get an in-between reaction of consumption.

We can see some of these implications even more clearly if we make stronger assumptions.

Instead of just assuming that $\beta(1+r) = 1$, let's furthermore assume that $r = 0$.⁵ This assumption is not so crazy – interest rates (and hence real interest rates) are pretty small numbers for the most part at most times. This assumption has the effect of simplifying the math significantly. The budget constraint becomes:

$$\sum_{j=0}^T C_{t+j} = \sum_{j=0}^T Y_{t+j}$$

This just says that the sum of consumption must equal the sum of income. If consumption must be the same in every period, which comes from the optimality condition plus the assumption that $\beta(1+r) = 1$, then the left hand side here just becomes:

$$\bar{C}(T+1) = \sum_{j=0}^T Y_{t+j}$$

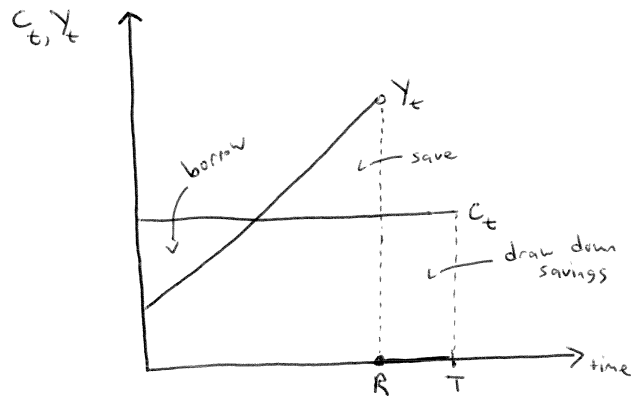
This follows because the sum of a constant over $T+1$ periods is just $T+1$ times the constant. Simplifying, we get:

$$\bar{C} = \frac{\sum_{j=0}^T Y_{t+j}}{T+1}$$

What is on the right hand side? That's just the sum of income divided by the number of periods, which is just average income. In other words, a simple way to think about the implications of the model is that consumption ought to be constant in each period of life and set equal to average lifetime income. If households live for a long time (T is big), then any transitory change in income will have a relatively small effect on average lifetime income, and hence a small effect on consumption. Changes in income which are persistent, in contrast, will have a large effect on average lifetime income, and hence a large effect on consumption.

This has implications for the “life cycle” of income and consumption. Our theory suggests that households ought to “smooth” their consumption, with consumption more or less constant throughout life. For the average person, income grows over time – it starts out low, grows during prime working years, and then goes down in retirement. The theory suggests that this predictable time path of income shouldn't have much effect on consumption. This means that households ought to borrow when they are young, save during their prime working years, and draw down their savings during retirement. The picture below plots this out:

⁵Do note that this is not quite consistent with all of our assumption. In particular, this could only hold if $\beta = 1$. Nevertheless, it makes the math work out nicely, so we'll proceed.



The above picture shows what the “life cycle” of income and consumption ought to look like. Consumption ought to be “smooth” and roughly constant. Consumption should be high early in life, even when current income is relatively low, if future income is expected to be relatively high. Financing this high consumption early in life requires borrowing – this is something many people do, through things like student loans, for example. In prime working years, when income is near its peak, saving should be positive. This positive saving pays off the debt from early in life and provides for the necessary stock of savings to finance consumption in retirement. When people retire, which occurs here at date R , income declines. Here I show it going all the way to zero; in reality it shouldn’t go all the way to zero because of things like social security and pensions. The important point, however, is that consumption should not decline. Under the assumptions we’ve made, consumption should be constant across life, and hence it has to be as high in retirement as it was during the prime working years. We can of course alter the assumptions to break that strict prediction, but something close to this is going to come out of any model of optimizing consumer behavior: consumption ought to be relatively smooth, and hence there should not be a precipitous drop in consumption at retirement.

5 Empirical Evidence and Extensions

In these notes we have developed a theory of consumer behavior. That theory – either in the two period or multi-period framework – makes a number of predictions that we can test in the data. The basic theory laid out goes by the name of the “Life Cycle / Permanent Income Hypothesis.” Some of its predictions are:

1. Consumption ought to be forward-looking, depending not just on current income but future income as well
2. Consumption ought to react more to permanent changes in income than transitory changes

in income

3. Current consumption should not react much to predictable changes in income that were anticipated in the past

A good way to summarize these predictions is that people would like their consumption to be “smooth.” The desire for smoothness is driven by the assuming that utility is concave – in words, that marginal utility is decreasing. If people are forward-looking and have this smoothing desire, then current consumption should depend on more than just current income – it should depend on expected future income. Below I talk briefly about some empirical evidence on these predictions, and then we wrap up with a discussion of modifications to the model that make it better fit the data in some dimensions.

There are several examples of tax cuts around the time of recessions in the last decade or so. In 2001, there were tax “rebates” sent to households, which amounted to roughly \$300 or \$600, depending on the filing status of an individual taxpayer. This was the first installment of the “Bush tax cuts,” which were set to last for ten years (there have been recent debates about whether to extend these tax cuts). Since they were to last for ten years, these tax cuts were more or less perceived to be “permanent,” and our theory would therefore predict that households should have increased their consumption significantly. In contrast to the theory, few households reported that they planned to spend much of their rebate checks.⁶

There was another similar round of tax rebates in 2008, during the height of the recent financial crisis. This was structured fairly similarly to the 2001 rebates, but it was understood that the \$300 - \$600 rebate checks were one-time – i.e. they were clearly transitory. The theory of consumption we have developed here would suggest that the consumption response to these rebate checks should be small. As it turns out, most people reported that they did not plan to increase their spending much in response to this round of rebates, consistent with the implications of the theory.⁷

We haven’t explicitly mentioned the third testable prediction of the model listed above. It is a corollary of the first point – consumption is forward-looking, and so anticipated changes in future income should be incorporated into current consumption. This means that there should not be predictable movements in consumption in response to changes in income that were predictable in the past. There are several papers that look into this phenomenon, with somewhat mixed results. One neat example is the way that Social Security tax withholding works. You pay around 7 percent of your income into Social Security up until about \$110,000 in income, after which you no longer pay social security taxes. Because of the way withholding works, this means that, if you earn enough income, at some point in the year your take home pay jumps up. For example, for someone earning \$150,000 a year, their take home pay ought to be roughly 7 percent higher starting sometime in the fall. Because this is a known feature of the tax code, consumption should not jump up in the

⁶See Matthew D. Shapiro and Joel Slemrod, “Consumer Responses to Tax Rebates,” *American Economic Review* 93, 381-396.

⁷See Matthew D. Shapiro and Joel Slemrod, “Did the 2008 Tax Rebates Stimulate Spending?” *American Economic Review: Papers & Proceedings* 99, 374-379.

fall when the paycheck jumps up – consumption early in the year should have been anticipating the increase in take home pay later in the year. In the data, consumption actually does rise when take-home pay predictably rises – not enormously, but pretty significantly nonetheless (about 20 cents on the dollar).⁸ This represents a failure of the theory.

Another good example of a predictable change in income is retirement. Households more or less know the date at which they retire (usually around age 65) and have their entire lives to plan for this event. Income predictably drops at retirement for most households – wage and salary income goes away, though most households still have some income from investments, pensions, and Social Security. To the extent to which the date of retirement is planned and known, our theory of consumption predicts that consumption should not drop at retirement – households should have been saving during their working years to prepare for retirement. Some studies have documented that measured expenditure on consumption goods does in fact drop at retirement, suggesting a failure of the theory. In an interesting paper, Hurst and Aguilar (2005) demonstrate that while expenditure on consumption does indeed drop, caloric intake does not. With more time on their hands, retired people (i) spend more time on meals at home as opposed to meals outside the house, which tends to reduce expenditure, and (ii) spend more time clipping coupons or shopping for best deals. Once one looks at actual consumption and not expenditure, the data are loosely consistent with the theory – there is no large drop in consumption at retirement.⁹

There is a well-known empirical fact that death rates in the US rise right after the first of the month. This pattern holds in almost every month – you are much more likely to die in the first three days of a month than in the last three days. For salaried workers in the US, there are typically two payment schemes: bimonthly and monthly. For those receiving a monthly paycheck, this check usually arrives either on the last day of the month or the first day of the month. Evans and Moore (2012) investigate the hypothesis of whether the increase in mortality is due to increased economic activity – more consumption leads to more driving and hence more driving fatalities, more drug use and therefore more overdoses, and heightened heart attack risk. The notion is that income receipt leads people to consume more, and that more consumption is responsible for the heightened mortality risk. This should not happen under our theory of consumption – you know you are going to get paid on the first of the month, and your consumption plans should incorporate this. These authors show convincingly that consumption responding to this predictable variation in income drives an important part of the within-month mortality cycle.¹⁰

Though there are some instances in which the predictions of the theory are borne out, it is clear that, on balance, the life cycle / permanent income hypothesis is rejected in the data. This means that the theory is not a perfect description of actual behavior and must be missing something. Note, however, that just because a theory does not hold perfectly in the data does not mean it is not useful for providing intuition and thinking about policy proposals. Any good theory is false –

⁸See Jonathan Parker, “The Reaction of Household Consumption to Predictable Changes in Social Security Taxes” *American Economic Review* 89, 959-973.

⁹See Aguilar and Hurst, “Consumption versus Expenditure,” *Journal of Political Economy* 113, 919-948.

¹⁰See Evans and Moore, “Liquidity, Activity, and Mortality,” *Review of Economics and Statistics* 94, 400-418.

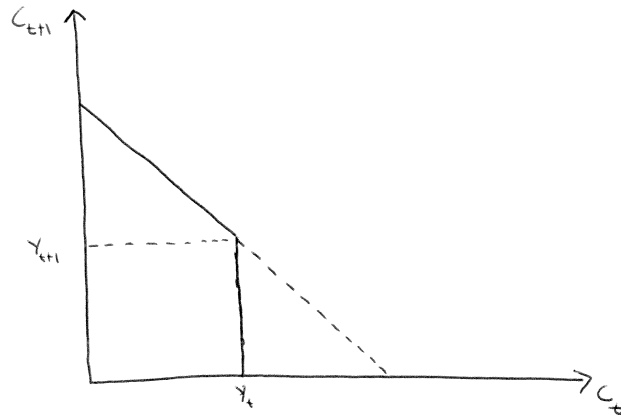
the theory must be an abstraction of a complex reality for it to be very useful.

Other than ignoring them, there are two basic ways of dealing with the failures of the theory. One highlights that there is some cost to optimization, and that households do not re-optimize following every change in income. I think there is a lot of appeal in this story. If I find an extra \$20 lying on the street, the theory says that I should increase my consumption by a tiny little bit in every period, saving most of it. But if I just spend the extra \$20 immediately, my utility loss is not that large. There is some mental cost to re-calibrating my consumption/saving plans, and hence, for small changes in income, it may not make sense to re-optimize – the utility cost to doing so exceeds the cost of re-doing the optimization.

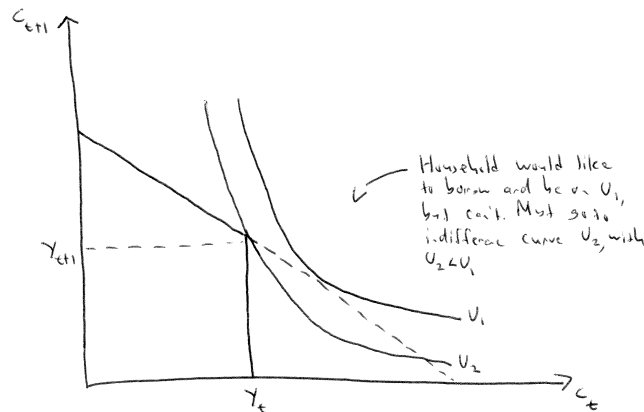
The other option is to modify the model in some way. The most realistic way of doing this is to introduce liquidity constraints. The predictions of the life cycle / permanent income hypothesis rest on the assumption that households can freely borrow and lend. For example, if you know that your income will go up in the future, the theory says that you should adjust your consumption immediately. Doing so requires that you be able to borrow in the present (or that you have a stock of savings built up which you can draw down). In reality for many people this is not possible or is prohibitively expensive (think about what interest rates on credit cards look like). Hence, even though they'd like to borrow, they can't, current consumption cannot react to a predicted future change in income, and you may observe consumption reacting to predictable movements in income in the period in which income actually changes. Also, if households are liquidity constrained, current consumption will react significantly more to transitory fluctuations in income than the baseline theory predicts.

A liquidity constraint can take on different forms. The gist of the idea is that households are inhibited in their ability to borrow. It is easiest to think about this in the context of the two period consumption model. An extreme version of a liquidity constraint is that the household can do positive saving and earn interest r_t , but is prohibited from borrowing.¹¹ That is, $S_t \geq 0$, which in turn implies that $C_t \leq Y_t$. This has the effect of introducing a “kink” into the budget constraint at the endowment point. The figure below shows a budget constraint with a kink like this. The dashed line shows what the budget constraint would look like without the kink:

¹¹A less extreme version is that the interest rate on borrowing exceeds the interest rate on saving. We can think about the extreme example here as one where the interest rate on borrowing is infinite.



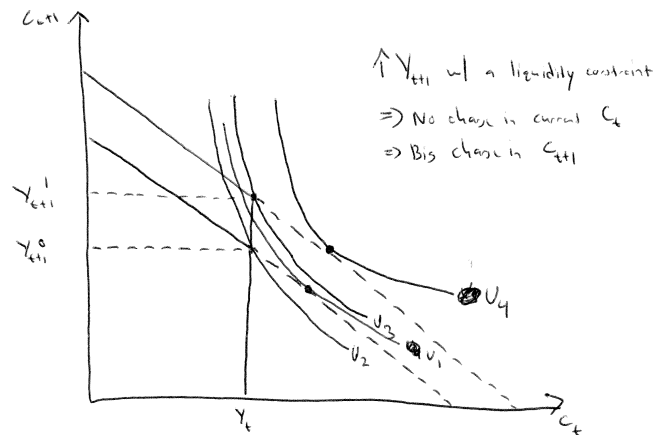
A liquidity constraint may or may not be “binding.” By binding I mean that the kink may have no effect on the household’s decision-making. You can tell whether the constraint would be binding by drawing in the hypothetical budget line without a constraint (the dashed line) and graphically finding the optimal consumption bundle. If the point where the indifference curve is tangent to the budget line is to the left of the kink, then the liquidity constraint is not binding. If the point where the indifference curve is tangent to the budget line is to the right of the kink (on the dashed component of the hypothetical budget line), then the constraint is binding – the household cannot achieve this point because of the constraint. We can see the effect of a binding liquidity constraint in the indifference curve-budget line diagram below:



In the above diagram the household would like to be on indifference curve U_1 where $C_t > Y_t$. However, because of the constraint that $S_t \geq 0$, this is not possible. The best the household can do is to locate on the kink, which is on indifference curve U_2 , with $U_2 < U_1$. Note that at the kink, the

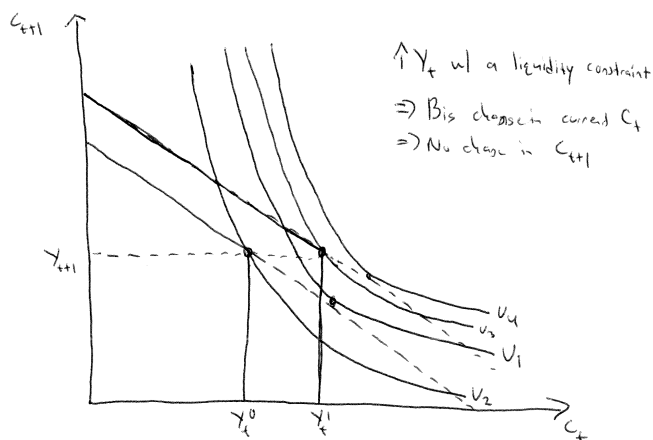
mathematical tangency condition will not hold. Relative to the optimality condition on U_1 with the hypothetical budget line, first period consumption is too small and second period consumption is too big, which means that $u'(C_t) > \beta(1+r_t)u'(C_{t+1})$, which follows because $u'(C_t)$ is decreasing in consumption.

We can use the indifference curve-budget line diagram with a binding liquidity constraint to think about two of the empirical failures of the life cycle permanent income hypothesis. First, suppose that there is a known increase in future income, Y_{t+1} . This would shift out the entire budget line in a world without constraints. In the world with constraints, it shifts out the upper portion of the budget line, but the kink still occurs at current income, which is unchanged. In a world without the constraint, the household would like to move from the optimality point on indifference curve U_1 to the point on U_4 . However, because of the constraint, the best it can do is to move from indifference curve U_2 (where $U_2 < U_1$) to U_3 (where $U_3 < U_4$), with the optimal consumption bundle moving from the location of the old kink to the new kink. We can see that there is no effect on current consumption, with future consumption reacting a lot. This is compared to the case if there were no constraint, where current consumption would increase and future consumption would increase, but by less than in the case where the constraint is binding. This setting thus offers a potential explanation for why, empirically, consumption appears to over-respond to predictable changes in income. If people are liquidity constrained, they would like to adjust their consumption immediately to anticipated changes in future income, but are unable to do so because of the constraint. They therefore end up spending the income when it arrives, as opposed to smoothing their consumption the way the theory predicts.



Next, consider a different experiment – a change in current income. As before, we begin in a situation in which the household is constrained – it would like to locate on indifference curve U_1 with $C_t > Y_t$, but cannot because of the constraint. The best it can do is to locate at the kink, which occurs at the endowment point. The increase in current income would have the effect of shifting the entire budget line out to the right. For regions above the kink, this is what happens.

The kink itself must also shift to the right, because the new endowment point has shifted to the right. But regions with C_t greater than the new level of current income remain unattainable. What the household would like to do is to move from indifference curve U_1 to U_4 , increasing both current and future consumption by a little. But it still can't reach U_4 even after the change in current income – the best it can do is to locate at the new kink point, associated with indifference curve U_3 , with $U_3 < U_4$. This means that current consumption will increase by the full amount of the increase in current income, with future consumption not moving at all.



We have thus seen that liquidity constraints can help us to understand two empirical failings of the life cycle / permanent income hypothesis model. In particular, binding liquidity constraints can explain why consumption over-reacts to predictable changes in income – the household would like to smooth its consumption and incorporate knowledge about a future change in consumption, but this is not possible because the household is not permitted to borrow against its future income. Secondly, a binding liquidity constraint can help us understand why consumption over-reacts to transitory changes in income. In a world without the constraint, the household would only increase current consumption by a fraction of a change in current income. But if a household would like to have current consumption greater than current income and is prohibited from borrowing, then a change in current income will lead to a much larger change in current consumption than the standard theory would predict. In particular, the “marginal propensity to consume” may well be 1 or very close to 1.

Liquidity constraints have potentially important implications for policy. I mentioned above examples of temporary tax cuts/rebates. The objective of these kinds of stimuli is to get people to consume more immediately, the hope being that people buying more stuff will lead to more employment, and hence less unemployment. Our theory suggests that consumption ought not to react strongly to a change in after tax income if that change is perceived as temporary. One way to encourage a stronger consumption response would be to make the tax cuts more persistent, but this may not always be feasible given concerns about the government budget deficit. The discussion of

liquidity constraints suggests, however, that there may be another way out – if you target temporary tax cuts at households who are likely to be liquidity constrained, the consumption response could be quite large. Households that are more likely to be liquidity constrained are poorer, younger, and generally less educated. Targeting tax rebates to households fitting this bill may have significantly larger consumption effects than the standard theory would predict.